Dynamic Price Competition with Local Network Effects Online Appendix

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1 Equilibrium Market Structure

Proposition 1. *For any value of $\delta \in [0, 1)$, there exist $\psi_1 \leq \psi_2 \leq \psi_3$ such that the equilibrium market structure is*

$$\begin{cases} \textit{Unimodal (Region A)} & \text{if } \psi < \psi_1, \\ \textit{Bimodal (Global dominance)(RegionB)} & \text{if } \psi_1 < \psi < \psi_2, \\ \textit{Four Modal (Region C)} & \text{if } \psi_2 < \psi < \psi_3, \\ \textit{Bimodal (Local dominance)(RegionD)} & \text{if } \psi > \psi_3 \end{cases}$$

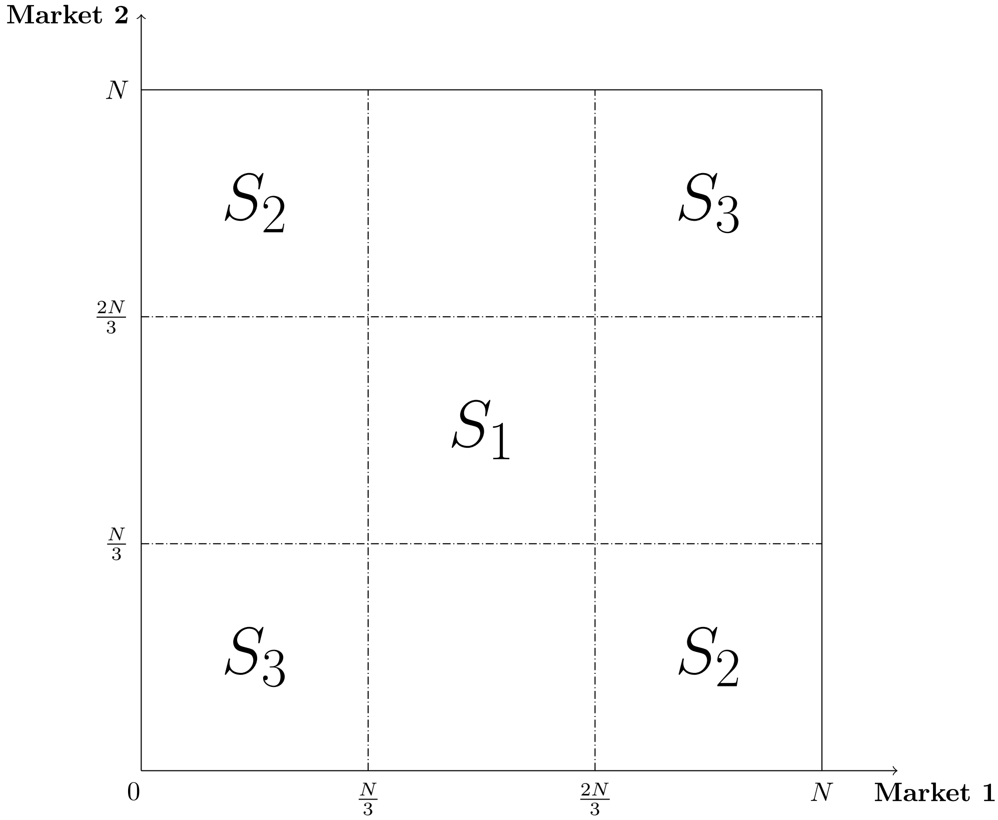

Figure 1: Equilibrium Market Structure

The numerical simulation of the model produces an equilibrium market share matrix for each combination of parameter values δ and *psi*. The matrix is shown in Figure 1. Then the matrix is divided into 9 regions and I sum up all the market share for each region and focus on three regions S_1, S_2, S_3.[1]

To summarize the result, each equilibrium market structure is categorized into one of the following groups: Unimodal(S_1), Bimodal (Local dominance)(S_2), Bimodal (Global dominance)(S_3), Four Modal(combination of S_2 and (S_3)).[2]

[1] The symmetric model produces symmetric equilibrium market structure, i.e. the two regions of S_2 are equal, and so are the S_3 regions.
[2] See Figure 6 for parameter values that produce each of these four equilibria

	Equilibrium Market Structure
$\mathbf{S_1 > S_2, S_3}$	Unimodal
$\mathbf{S_2 > S_1, S_3}$	
$1 < S_2/S_3 < 1.5$	Four Modal
$S_2/S_3 > 1.5$	Bimodal (Local Dominance)
$\mathbf{S_3 > S_1, S_2}$	
$1 < S_3/S_2 < 1.5$	Four Modal
$S_3/S_2 > 1.5$	Bimodal (Global Dominance)

Table 1: Categorization Rules

The categorization rules used to produce Figure 4 is shown in Table 1.

2 Pricing Function

In the main text of this paper, the firm's optimal pricing strategy is characterized as follows.

$$p_i = h(i_1, i_2) - w(i_1, i_2) \tag{1}$$

where

$$h(i_1, i_2) = -\frac{q(i_1, i_2) + q(i_2, i_1)}{q'(i_1, i_2) + q'(i_2, i_1)} \tag{2}$$

$$w(i_1, i_2) = \frac{q'(i_1, i_2)}{q'(i_1, i_2) + q'(i_2, i_1)} \frac{1}{N} w_1(i_1, i_2) + \frac{q'(i_2, i_1)}{q'(i_1, i_2) + q'(i_2, i_1)} \frac{1}{N} w_2(i_1, i_2) \tag{3}$$

in which

$$\begin{aligned} w_1(i_1, i_2) = &-i_1\theta(i_1 - 1, i_2) + (i_1 - j_1)\theta(i_1, i_2) + j_1\theta(i_1 + 1, i_2) \\ &+ \delta[-i_1 v(i_1 - 1, i_2) + (i_1, j_1)v(i_1, i_2) + j_1 v(i_1 + 1, i_2)] \end{aligned} \tag{4}$$

$$\begin{aligned} w_2(i_1, i_2) = &-i_2\theta(i_1, i_2 - 1) + (i_2 - j_2)\theta(i_1, i_2) + j_2\theta(i_1, i_2 + 1) \\ &+ \delta[-i_2 v(i_1, i_2 - 1) + (i_2, j_2)v(i_1, i_2) + j_2 v(i_1, i_2 + 1)] \end{aligned} \tag{5}$$

2.1 The Harvesting Effect

As mentioned in the main text, $h(i_1, i_2)$ represents the harvesting effect that motivates the firm to exploit more consumer surplus by raising the price. It is also related to price elasticitiy as explained in the paper. Another simpler intuition can be drawn to show the connection between this harvesting effect and the price elasticity. From Cabral (2011) one can show that

$$h = -\frac{q_i}{q_i'} = \frac{1}{-\frac{q_i'}{q_i}} = \frac{1}{\epsilon} \tag{6}$$

$\epsilon = -\frac{q_i'}{q_i}$ is the price elasticity, in which q_i is the probability that a new consumer choosing network i (Demand) and q_i' is the derivative of the demand with respect to price p_i. So ϵ measures the changes in demand in response to a change in price. In my paper, demand elasticity also determines the harvesting effect:

$$h_{12} = -\frac{q_{12} + q_{21}}{q_{12}' + q_{21}'} = \frac{1}{-\frac{q_{12}' + q_{21}'}{q_{12} + q_{21}}} = \frac{1}{\epsilon} \tag{7}$$

So $\epsilon = -\frac{q_{12}' + q_{21}'}{q_{12} + q_{21}}$ is an weighted average of the harvesting effects in both local market 1 and 2. Here the population N is not in this expression because I assume the same population. In fact, in another version of this model with heterogeneous population N_1 and N_2 at each local market, the price elasticity depends on the relative number of population: $\epsilon = -\frac{N_1 q_{12}' + N_2 q_{21}'}{N_1 q_{12} + N_2 q_{21}}$. When one local market is larger than the other one, the firm puts less weight the smaller market; when one market has no population, the model converges to the model developed by Cabral (2011), where the network effects is global.

2.2 The Investing Effect

$w(i_1, i_2)$ shows the investing effect. It is expressed as the difference in the firm's benefit between attracting and losing a new customer. The total investing effect $w(i_1, i_2)$ is a weighted average of both local investing effects at each local market. w_1 and w_2, and each of the local investing effect includes the change of the firm's current benefit and future benefit by attacting a new customer.

The weights are calculated by the relative price sensitivity. If consumers in local market 1 are significantly more sensitive than those in market 2, the firm focuses more on consumers in market 1.[3]

Similar to the discussion of the harvesting effect above, another version of the model with heterogenous population gives

$$w(i_1, i_2) = \frac{N_1 q'(i_1, i_2)}{N_1 q'(i_1, i_2) + N_2 q'(i_2, i_1)} \frac{1}{N_1} w_1(i_1, i_2) + \frac{N_2 q'(i_2, i_1)}{N_2 q'(i_1, i_2) + N_1 q'(i_2, i_1)} \frac{1}{N_2} w_2(i_1, i_2) \tag{8}$$

In this case, firms adjust their pricing strategy by taking the market sizes into consideration.

To understand the evolution of different equilibrium path, I start with analyzing the platform choice of a new customer, which has three components as shown in equations (**??**)) and (**??**): 1. Price differences offered by both platforms. 2. Personal network preferences. 3. Expected future consumer values.

For industries with a slightly stronger network, e.g. $\psi = 0.12$, the stationary distribution of market shares shows an equilibrium path with global dominance, in which one of the platforms dominates in both market segments. In this case, both the small and large platform charge a high price. This is the case only when ψ is not too high, so that a certain proportion of new customers still choose the small network even though the dominant network has much larger user base. However, as the strength of network effects goes even higher, small platform cannot attract any customers by charging high price any more. Later I will show a detailed analysis why this is the case.

Industries with even stronger network, e.g. $\psi = 0.15$, the equilibrium market structure changes from global dominance (region B) to a mixture of global and local dominance (region C).

When network effects is strong enough in an industry, e.g. $\psi = 0.18$, customers place a very high value on existing user base, so very few new customers choose the smaller platform. The small platform cannot sustain a high-pricing strategy, and it has no choice but to fight for more market share through a price war against the dominant firm. Eventually the small platform becomes a dominant firm in one market (local dominance). We can gain more insight from the equilibrium values behind the graph when $\psi = 0.12$ and $\psi = 0.18$ shown in table 1.[4]

Table 1 provides a more complete equilibrium for low ($\psi = 0.12$ in region B) and high ($\psi = 0.18$ in region D) strength of network effects. They are calculated by dividing the state space (market share in segment A and B) into 9 regions, and calculate the average value within that region. Four of the values that are included in the table are:

1. **Probability of Equilibrium**: The probability of observing any one platform having each market share. e.g. When $\psi = 0.12$, the probability of any platform being the small platform in both segment is 18%; the probability of being large in segment A and small in segment B is 7%.

2. **Firm Value**: Discounted value of all future benefits and revenue each market share.

3. **Probability of a Sale**: For a platform with a certain market share, the probability of a new consumer from segment A buys from that platform. e.g. When $\psi = 0.12$, platform i is the large firm in segment A and smalll one in segment B. In equilibrium, the probability that a new customer from segment A buys from platform i is 74%.

4. **Equilibrium Prices**: Prices charged by a platform with a certain market share.

First, in an industry where network effects is not very strong, small platform is still chosen by some new customers. Table 1 shows that when $\psi = 0.12$, despite a higher price charged by the small platform (0.17 vs. 0.12), the small platform can still attract 23% of new customers. For example, in the tablet industry, consumers value both direct and indirect network effects. Many Android-tablet manufacturers, though having smaller market share, still charges similar prices as iPad, and some consumers are still willing to pay for it, mostly because of their preference of the Android system. However, if the network becomes stronger, i.e. consumers' value of network size becomes higher, more and more consumers choose the tablet with more user base, which is iPad, and Android tablets would not be able to afford charging a similar price any more. When this is the case, the smaller platform lives in the "comfort zone" of not fighting with the larger platform in order to gain more market share, because of the high cost of a long term price war.

Second, in an industry where consumers value network size strongly, few consumers join the small platform, unless for extremely lower price. When $\psi = 0.18$, all new consumers value highly of existing user base, and the small

[3]For example, $w_1 = 100$, $w_2 = 5$ means the additional value brought by an additional consumer from market 1 and market 2 are 100 and 5 respectively. By this measure alone, it seems that market 1 is a more profitable investment. But if $q'(i_1, i_2) = 0.01$ and $q'(i_2, i_1) = 1$, meaning that consumers in local market 2 are much more price sensitive to those in market 1, then lowering prices in market 2 becomes a better choice.

[4]Note that represent the states of the model of the stationary distribution, i.e. states which one most likely observe in equilibrium.

ψ = 0.12

Probability of Equilibrium

Segment A \ Segment B	Small	Medium	Large
Small	18% *	13%	7%
Medium	13%	14%	13%
Large	7%	13%	18% *

Firm Value

Segment A \ Segment B	Small	Medium	Large
Small	5.38 *	13.25	30.59
Medium	13.25	22.9	40.61
Large	30.59	40.61	56.98 *

Probability of a Sale

Segment A \ Segment B	Small	Medium	Large
Small	23% *	24%	26%
Medium	46%	50%	54%
Large	74%	76%	77% *

Equilibrium Prices

Segment A \ Segment B	Small	Medium	Large
Small	0.17 *	-0.01	0.15
Medium	-0.01	-0.17	-0.04
Large	0.15	-0.04	0.12 *

ψ = 0.18

Probability of Equilibrium

Segment A \ Segment B	Small	Medium	Large
Small	0%	0%	50% *
Medium	0%	0%	0%
Large	50% *	0%	0%

Firm Value

Segment A \ Segment B	Small	Medium	Large
Small	2.95	29.88	78.62 *
Medium	29.88	32.77	61.26
Large	78.62 *	61.26	108.14

Probability of a Sale

Segment A \ Segment B	Small	Medium	Large
Small	12%	13%	5% *
Medium	74%	50%	26%
Large	95% *	87%	88%

Equilibrium Prices

Segment A \ Segment B	Small	Medium	Large
Small	0.08	-0.2	2.81 *
Medium	-0.2	-0.77	-0.01
Large	2.81 *	-0.01	0.26

Table 2: Comparative Statics of Equilibrium ($\delta = 0.92$)

platform's high price strategy cannot attract as many new customers as in the case with $\psi = 0.12$. The best response of the small platform is to reduce its price dramatically in order to gain a market segment. Note that when $\psi = 0.12$ the small platform is not willing to fight, but it fights because that's the only way to attract new customers. Gradually, the dominant platform gives up one market segment to the small platform in order to avoid a long term price war.

Third, when the network is strong, platforms "collude" by focusing on their own segment. When $\psi = 0.18$, both platforms are charging 2.81, much higher than what they would charge under any other values of state variables, and both attract 95% of the new customers in their own segment.

Fourth, when the network is not strong, platforms are unable to "collude" with local dominance. When $\psi = 0.12$ and each platform dominates in one segment, both would like to start a price war and become a dominant player in both segments. On one hand, the potential gain in firm value from local dominance to global dominance is large ($(56.98 - 30.59)/30.59 = 86\%$); on the other hand, the losing firm goes back to the "comfort zone", which is not too worse off (from 30.59 to 5.38). However, in the case with strong network, the potential gain from local to global dominance is relatively small ($(108.14 - 78.62)/78.62 = 37\%$), and the lost is much larger than in previous case (from 78.62 to 2.95).

3 Analytical Results

Proposition 2. *For $\psi = 0$ and any $\delta \in [0, 1)$, the equilibrium prices p_i and p_j are constant.*

When $\psi = 0$, the game is essentially reduced to the Hotelling duopoly with consumer distribution $\phi(x)$. Two firms offer their prices, and the consumer choice is purely based on the network preference and the price difference because there is no network effects. In equilibrium, both firms offer the same price, and $q = 1/2$, meaning that a new consumer chooses network 1 or 2 with equal probability.

When there are no network effects, the size of a network is not valued by neither consumers nor firms. Any dynamic effects disappear without any connection across different time periods, and all equilibrium results remain unchanged throughout time. Maximizing firm value is equivalent to maximizing current profit. So there is no investing effect through having a larger user base. One real life example of competition without network effects is between Coca Cola and Pepsi Cola. Both firms set the same price, and roughly half of the consumers choose Coca Cola, and the other half choose Pepsi Cola.

Proof of Proposition 1. When $\psi = 0$, both $\theta(i_A, i_B) = 0$ and $\lambda(i_A) = 0$. According to the updating function of consumer utility equation (5) and the position of the indifference consumer equation (1) and (2), we can easily get $x(i_A, i_B) = x(i_B, i_A)$. Therefore, from the probability of a sale in equation (3), we can also have $q(i_A, i_B) = q(i_B, i_A)$. Next we will look into p_i and p_j:

$$p_i = \frac{1 - \Phi(x(i_A, i_B))}{\phi(x(i_A, i_B))} - \frac{1}{2}w_1(i_A, i_B) - \frac{1}{2}w_2(i_A, i_B)$$

$$\begin{aligned} p_i &= \frac{(1 - \Phi(x(j_A, j_B))}{\phi(x(i_A, i_B))(x(j_A, j_B)} - \frac{1}{2}w_1(j_A, j_B) - \frac{1}{2}w_2(j_A, j_B) \\ &= \frac{\Phi(x(i_A, i_B))}{\phi(x(i_A, i_B))} - \frac{1}{2}w_1(j_A, j_B) - \frac{1}{2}w_2(j_A, j_B) \end{aligned}$$

Therefore, we can get

$$p_i - p_j = \frac{(1 - 2\Phi(x(i_A, i_B))}{\phi(x(i_A, i_B))} - \frac{1}{2}(w_1(j_A, j_B) + w_2(j_A, j_B) - w_1(i_A, i_B) - w_2(i_A, i_B))$$

$$\begin{aligned} \text{where } w_1(i_A, i_B) &= \delta(-i_A v(i_A - 1, i_B) + (i_A - j_A)v(i_A, i_B) + j_A v(i_A + 1, i_B)) \\ w_1(j_A, j_B) &= \delta(-j_A v(j_A - 1, j_B) + (j_A - i_A)v(j_A, j_B) + i_A v(j_A + 1, j_B) \\ w_2(i_A, i_B) &= \delta(-i_B v(i_A, i_B - 1) + (i_B - j_B)v(i_A, i_B) + j_B v(i_A, i_B + 1)) \\ w_2(j_A, j_B) &= \delta(-j_B v(j_A, j_B - 1) + (j_B - i_B)v(j_A, j_B) + i_B v(j_A, j_B + 1) \end{aligned}$$

$$\begin{aligned} p_i - p_j + \frac{2\Phi(x(i_A, i_B)) - 1}{\phi(x(i_A, i_B))} &= -\frac{1}{2}(w_1(j_A, j_B) + w_2(j_A, j_B) - w_1(i_A, i_B) - w_2(i_A, i_B)) \\ p_i - p_j + \frac{2\Phi(p_i - p_j + \frac{1}{N}f_1(u)) - 1}{\phi(p_i - p_j + \frac{1}{N}f_1(u))} &= -\frac{1}{2}(w_1(j_A, j_B) + w_2(j_A, j_B) - w_1(i_A, i_B) - w_2(i_A, i_B)) \end{aligned}$$

Here we can get the existence of the solution by taking a guess and check the whether it is correct by updating it. By doing so the solution is guaranteed to exist although the uniqueness is not proved. The uniqueness can be shown in numerical simulation. We guess that $v(i_A, i_B)$ is constant across $i_A, i_B \in [0, N]$, thus according to equations (10) and (11), we can get $w_1(i_A, i_B), w_2(i_A, i_B), w_1(j_A, j_B), w_2(j_A, j_B) = 0$.

Then we have

$$p_i - p_j = \frac{1 - 2\Phi(p_i - p_j)}{\phi(p_i - p_j)} \tag{9}$$

As the left-hand side $p_i - p_j$ increases, the right-hand side will decrease according to the assumption 1 (iv). Therefore, we can get a unique value of $p_i - p_j$, which is 0 here. Therefore, we can also get $x(i_A, i_B) = 0$ and $q12 = \frac{1}{2}$. Moreover, $p_i = h_i = \frac{1-\Phi(0)}{\phi(0)}$, which is a constant given any distribution that satisfies assumption 1, and it does not change with different values of δ.

Moreover, By solving the systems of equations of the firm value functions (13) we can also get equilibrium firm value $v(i_A, i_B) = \frac{1}{4\phi(0)(1-\delta)}$ □

Proposition 3. *There exists a δ' such that, if $\delta < \delta'$, (a) if $\theta(i_A + 1, i_B) - \theta(i_A, i_B)$ and $\theta(i_A, i_B + 1) - \theta(i_A, i_B)$ are constant, and Property* **??** *holds strictly, then $p_i(i_A, i_B)$ is strictly incerasing in i_A and i_B, and $p_j(j_A, j_B)$ is strictly incerasing in j_A and j_B. (b) if $\lambda(i_A)$ and $\lambda(i_B)$ are constant and Property* **??** *holds strictly, then $p_i(i_A, i_B)$ is strictly decreasing in i_A and i_B, and $p_j(j_A, j_B)$ is strictly decreasing in j_A and j_B.*

Proposition 3 provides more insights on how platforms make the pricing decisions, and it highlights the two main factors harvesting and investing effects mentioned before. In case (a), platforms' aftermarket per-period benefit from network effects increases linearly, i.e. constant return to scale. Larger network do not benefit more from gaining a new customer than smaller network. The pricing incentive comes only from the network effects enjoyed by consumers, who value larger customer base. Therefore, when the customer base of one platform in one market segment grows, that platform has incentive to increase its entry price to extract more consumer surplus from new consumers.

On the other hand, in case (b), consumers do not enjoy network effects, and the only source of network effects comes from platforms' aftermarket benefits. Larger platform benefits more from gaining a new sale compared to smaller platform does, so the larger platform lowers its price in order to attract new customers. When the customer base of one platform in the one market segment increases, that platform has incentive to decrease its price in order to attract new customers and enjoy a greater aftermarket benefits.

Proof of Proposition 3. (a) Let $\theta(i_A+1, i_B) - \theta(i_A, i_B) = \theta(i_A, i_B+1) - \theta(i_A, i_B) = k$, which is a positive constant value.

From the proof of Proposition **??**, we have

$$\Delta p_i + \frac{\Phi_{AB}+\Phi_{BA}}{\phi_{AB}+\phi_{BA}} - \frac{\Phi_{-AB}+\Phi_{-BA}}{\phi_{-AB}+\phi_{-BA}} = \frac{\phi_{AB}}{\phi_{AB}+\phi_{BA}}(w_1^j - w_1^i) + \frac{\phi_{BA}}{\phi_{AB}+\phi_{BA}}(w_2^j - w_2^i)$$

At $\delta = 0$,

$$w_1^i \equiv w_1(i_A, i_B) = \frac{i_A}{N}(\theta_{i_A,i_B} - \theta_{i_A-1,i_B}) + \frac{j_A}{N}(\theta_{i_A+1,i_B} - \theta_{i_A,i_B}) = \frac{i_A+j_A}{N}k = k$$
$$w_1^j \equiv w_1(j_A, j_B) = \frac{j_A}{N}(\theta_{j_A,j_B} - \theta_{j_A-1,j_B}) + \frac{i_A}{N}(\theta_{j_A+1,j_B} - \theta_{j_A,j_B}) = \frac{i_B+j_B}{N}k = k$$

Thus $w_1^j - w_1^i = 0$. Similarly $w_2^j - w_2^i = 0$. Then we get:

$$\Delta p_i + \frac{\Phi_{AB}+\Phi_{BA}}{\phi_{AB}+\phi_{BA}} - \frac{\Phi_{-AB}+\Phi_{-BA}}{\phi_{-AB}+\phi_{-BA}} = 0 \tag{10}$$
$$\text{in which } \Phi_{AB} \equiv \Phi(x_{AB}); \Phi_{-AB} \equiv \Phi_(-x_{AB})$$
$$x_{AB} = \Delta p_i - \Delta Eu_{AB}$$
$$x_{BA} = \Delta p_i - \Delta Eu_{BA}$$
$$\Delta Eu_{AB} = \frac{1}{N}[j_A\lambda_{j_A} + i_A\lambda_{j_A+1} - i_A\lambda_{i_A} - j_A\lambda_{i_A+1}]$$
$$= \frac{1}{N}[i_A(\lambda_{j_A+1} - \lambda_{i_A}) + j_A(\lambda_{j_A} - \lambda_{i_A+1})]$$
$$\Delta Eu_{BA} = \frac{1}{N}[j_B\lambda_{j_B} + i_B\lambda_{j_B+1} - i_B\lambda_{i_B} - j_B\lambda_{i_B+1}]$$
$$= \frac{1}{N}[i_B(\lambda_{j_B+1} - \lambda_{i_B}) + j_B(\lambda_{j_B} - \lambda_{i_B+1})]$$

According to Property 1, Eu_{AB} increases in i_A but not affected by i_B and Eu_{BA} increases in i_B but not affected by i_A. To prove the final result, we still need to know the relationship between i_A and x_{AB}, as well as i_B and x_{BA}, shown in the following lemma:

Lemma 1. *x_{AB} decreases in i_A, and x_{BA} decreases in i_B.*

Proof. To show the first part of the Lemma 1, we subtract Eu_{AB} from both sides of equation (10):

$$\Delta p_i - Eu_{AB} + \frac{\Phi_{AB}+\Phi_{BA}}{\phi_{AB}+\phi_{BA}} - \frac{\Phi_{-AB}+\Phi_{-BA}}{\phi_{-AB}+\phi_{-BA}} = -Eu_{AB}$$
$$x_{AB} + \frac{\Phi(x_{AB})+\Phi(x_{BA})}{\phi(x_{AB})+\phi(x_{BA})} - \frac{\Phi(-x_{AB})+\Phi(-x_{BA})}{\phi(-x_{AB})+\phi(-x_{BA})} = -Eu_{AB} \tag{11}$$

Previously, we have established that Eu_{AB} increases in i_A only and Eu_{BA} increases in i_B only. According to Assumption (iv) we also know that the value of the left-hand side of the equation above is increasing in x_{AB}. So when i_A increases, x_{BA} does not change and the value of the right-hand side of equation (11) decreases, therefore, we conclude conclude that x_{AB} decreases. The same steps can be used to prove the second part of this lemma. □

Finally, going back to the original pricing function:

$$p_i = \frac{\Phi(-x_{AB})+\Phi(-x_{BA})}{\phi(-x_{AB})+\phi(-x_{BA})} - \frac{\phi_{AB}}{\phi_{AB}+\phi_{BA}}w_1^i - \frac{\phi_{BA}}{\phi_{AB}+\phi_{BA}}w_2^i$$
$$= \frac{\Phi(-x_{AB})+\Phi(-x_{BA})}{\phi(-x_{AB})+\phi(-x_{BA})} - k \tag{12}$$

So when i_A increases, Lemma 1 showed x_{AB} decreases, and the first term of right-hand side of equation (12)) decreases in x_{AB}. Therefore, we conclude that p_i increases in i_A. The same steps can also be used to prove that p_i increases in i_B. □

Proof of Proposition 3 (b).

$$p_i = h(i_A, i_B) - w(i_A, i_B) \tag{13}$$

In order to show that p_i decreases in i_A and i_B, we can divide the proof into two lemmas.

Lemma 2. *$w(i_A, i_B)$ increase in i_A and i_B.*

Proof.

$$\begin{aligned}
w(i_A, i_B) &= \frac{\phi(x(i_A, i_B))}{\phi(x(i_A, i_B)) + \phi(x(i_B, i_A))} w_1(i_A, i_B) + \frac{\Phi(x(i_B, i_A))}{\phi(x(i_A, i_B)) + \phi(x(i_B, i_A))} w_2(j_A, j_B) \\
\text{in which } w_1(i_A, i_B) &= \frac{i_A}{N}(\theta_{i_A,i_B} - \theta_{i_A-1,i_B}) + \frac{j_A}{N}(\theta_{i_A+1,i_B} - \theta_{i_A,i_B}) \\
w_2(i_A, i_B) &= \frac{i_B}{N}(\theta_{i_A,i_B} - \theta_{i_A,i_B-1}) + \frac{i_B}{N}(\theta_{i_A,i_B+1} - \theta_{i_A,i_B}) \\
x_{AB} &= \Delta p_i - \Delta E u_{AB} \\
&= \Delta p_i + \frac{1}{N}[j_A\lambda_{j_A} + i_A\lambda_{j_A+1} - i_A\lambda_{i_A} - j_A\lambda_{i_A+1}] \\
&= \Delta p_i + 0 \text{ (since } \lambda(i_A) \text{ is constant)} \\
&= \Delta p_i \\
x_{AB} &= \Delta p_i
\end{aligned}$$

Thus, $w_1(i_B, i_A) = \frac{1}{2}(w_1(i_A, i_B) + w_2(j_A, j_B))$. According to Property **??**, we can also show that $w_1(i_A, i_B)$ increases in i_A:

$$\begin{aligned}
& w_1(i_A+1, i_B) - w_1(i_A, i_B) \\
=& \frac{i_A+1}{N}(\theta_{i_A+1,i_B} - \theta_{i_A,i_B}) + \frac{j_A-1}{N}(\theta_{i_A+2,i_B} - \theta_{i_A+1,i_B}) \\
& \quad - \frac{i_A}{N}(\theta_{i_A,i_B} - \theta_{i_A-1,i_B}) - \frac{j_A}{N}(\theta_{i_A+1,i_B} - \theta_{i_A,i_B}) \\
=& \frac{i_A}{N}(\theta_{i_A+1,i_B} - \theta_{i_A,i_B} - (\theta_{i_A,i_B} - \theta_{i_A-1,i_B})) + \frac{j_A}{N}(\theta_{i_A+2,i_B} - \theta_{i_A+1,i_B} - (\theta_{i_A+1,i_B} - \theta_{i_A,i_B})) \\
& \quad + \frac{1}{N}(\theta_{i_A+1,i_B} - \theta_{i_A,i_B} - (\theta_{i_A+2,i_B} - \theta_{i_A+1,i_B})) \\
=& \frac{i_A}{N}(\theta_{i_A+1,i_B} - \theta_{i_A,i_B} - (\theta_{i_A,i_B} - \theta_{i_A,i_B})) \\
& \quad + \frac{j_A-1}{N}(\theta_{i_A+2,i_B} - \theta_{i_A+1,i_B} - (\theta_{i_A+1,i_B} - \theta_{i_A,i_B})) \\
& > 0 \qquad \text{(according to Property ??)}
\end{aligned}$$

$w_2(i_A, i_B)$ increases in i_B can also be shown using the same logic. Therefore, we can conclude that $w(i_A, i_B)$ increase in i_A and i_B. □

Lemma 3. *p_i decreases in $w(i_A, i_B)$.*

Proof. We know from **??** that $h(i_A, i_B) = \frac{2-\Phi_{AB}-\Phi_{BA}}{\phi_{AB}+\phi_{BA}}$. Define $H(i_A, i_B) \equiv h(i_A, i_B) - h(j_A, j_B) = \frac{2-\Phi_{AB}-\Phi_{BA}}{\phi_{AB}+\phi_{BA}}$, and $h'_i \equiv \frac{dh(i_A,i_B)}{dp_i}$, $H'_i \equiv \frac{dH(i_A,i_B)}{dp_i} < 0$, then we have:

$$\begin{aligned}
\frac{dp_i}{dw_i} =& \frac{dh_i}{dw_i} - 1 \\
=& \frac{dh_i}{dP_i}\frac{dP_i}{dw_i} - 1 \\
=& h'_i(-\frac{1}{1-H'_i}) - 1 \qquad \text{(from implicit function theorem)} \\
=& \frac{1}{2}H'_i(-\frac{1}{1-H'_i}) - 1 \\
<& 0 \qquad \text{which means that } p_i \text{ decreases in } w(i_A, i_B)
\end{aligned}$$

□

The proposition follows naturally from combining both lemmas. □

References